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Combined Approximations for Efficient Probabilistic Analysis of Structures

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Real-life analysis and design problems involve uncertainties. Quantification of the uncertainties in a system's response is important and requires a probabilistic analysis of the system. A main challenge in probabilistic analysis of large structural systems is the high computational effort due to the multiple repeated analyses involved. The combined approximations (CA) method, which combines the strengths of both local and global approximations, can be used for efficient probabilistic analysis of structures. The CA method is a combination of binomial series (local) approximations (also called Neumann expansion approximations) and reduced basis (global) approximations. An efficient method is presented for probabilistic analysis of structural systems using the CA method. The effectiveness of this method is demonstrated on analysis of mistuned bladed disk assemblies and systems with progressive collapse using Monte Carlo simulation. It is shown that the method can predict accurately the probability distribution function of the responses of these systems at a considerably lower cost than a method using finite element analysis in each cycle of Monte Carlo simulation.

I. Introduction

ALL engineering problems involve uncertainties including variability in the geometry, in the material properties, or in the loading. It is important to quantify these uncertainties and evaluate their effect on a system's response because uncertainties significantly affect performance. A probabilistic analysis calculates the probability density function of the system's response parameters and the reliability of the system. This is important information for making design decisions involving safety and cost tradeoffs.

Reliability assessment of a structural system is considerably more complex and expensive than that of a component. Moses[1] emphasized the complexities involved in probabilistic analysis of structural systems. Melchers[2] also addressed the issue of reliability prediction in structural systems. There are various computer codes for probabilistic analysis of components and systems including VeroSOLVE,[3] ProFES,[4] NESSUS,[5] and others. Most of the existing probabilistic codes are limited to application to simple static problems because they are based on approximate (first-order and second-order) reliability methods for a given (time-invariant) limit-state function.

Nikolaidis and Kapania[6] reviewed methods for evaluating system reliability and redundancy. They emphasized the importance of system reliability in conjunction with design optimization.

System reliability can be computed by using approximate methods, such as first-order or second-order reliability methods, or simulation methods, such as Monte Carlo. In many problems, Monte Carlo simulation can be more suitable for predicting the reliability of systems,[7] but it is also expensive. To make the computations tractable, response surface approximations are often employed. However, these approximations are effective only in problems with small numbers of variables, for example, fewer than 10–15.

Many engineering problems, such as those of finding the stresses in a structure, require solution of a linear set of equations. In probabilistic analysis of such systems, a linear set of equations needs to be solved hundreds or thousands of times. The computational cost is a principal obstacle in probabilistic analysis of large systems. The combined approximations (CA) method can be used for probabilistic analysis of structures. This method can be used to approximate the response of systems with many variables. The CA method provides good approximations even for large perturbations in the variables.

Kirsch[8] proposed the CA method, which is an efficient reanalysis method. The term combined approximations indicates that the method combines the advantages of both global and local approximations, that is, the approximation is accurate not only around a point corresponding to the nominal values of the design variables but also over a wide range of values of these variables. The CA method is efficient and easy to implement. It is also nonintrusive, that is, one can use it together with a finite element analysis program without changing the program.

The objectives of this paper are to 1) develop an efficient method for probabilistic analysis of structures using the CA method and 2) illustrate the efficiency and accuracy of this method on probabilistic analysis of real-life engineering problems.

The CA method is used for evaluating the forced response of bladed disk assemblies. The probability distribution of the blade amplitudes is calculated. The CA method is also used for finding the probability density function of the ultimate collapse load of truss structures with progressive failure modes.

First, the CA method is introduced. Then the method is illustrated on probabilistic analysis of mistuned bladed disk assemblies. Specifically, the problem of predicting the response of mistuned bladed disk assemblies, the solution procedures for evaluating the steady-state harmonic forced response of the blades using a direct analysis method and the CA method, and a numerical example are

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presented. The CA method is also illustrated on probabilistic analysis of progressive collapse of structures.

II. CA Method

Several studies developed approximations for structural optimization problems.[8–11] Initially, most of the approximations were valid only for small changes in a design. For large design changes, the accuracy of the approximations deteriorated.

Kirsch and Tolendano[12] outlined a number of approximate reanalysis methods for modifications in structural geometry. These include first- and second-order Taylor series approximations, polynomial fitting (quadratic and cubic), and simple iteration procedures based on series approximations.

Kirsch[13] proposed the CA method for structural optimization problems. Most of the literature pertaining to the development of the CA method consists of a series of papers published by Kirsch,[14–18] Kirsch and Papalambros,[19] Kirsch et al.,[20] and Kirsch.[21,22] CAs are a combination of binomial series approximations (also called Neumann expansion methods) (see Yamazaki et al.[7]) and reduced basis approximations. The effectiveness of CA method was demonstrated in topology optimization, vibration, nonlinear reanalysis of structures, and sensitivity analysis.[8]

In the following paragraphs the CA method will be explained for linear static analysis problems. The problem of static reanalysis can be stated as follows. Given an initial design with stiffness matrix $\boldsymbol{K}_0$ and load vector $\boldsymbol{R}_0$, the displacements $\boldsymbol{r}_0$ are computed by the initial equilibrium equations

$$\boldsymbol{K}_0 \boldsymbol{r}_0 = \boldsymbol{R}_0 \tag{1}$$

where $\boldsymbol{K}_0$ is given from the initial analysis in the decomposed form

$$\boldsymbol{K}_0 = \boldsymbol{U}_0^T \boldsymbol{U}_0 \tag{2}$$

and $\boldsymbol{U}_0$ is an upper triangular matrix. When a change is assumed in the design and corresponding changes $\Delta\boldsymbol{K}_0$ in the stiffness matrix and $\Delta\boldsymbol{R}_0$ in the load vector are assumed, the object is to estimate efficiently and accurately the modified displacements $\boldsymbol{r}$ without solving the complete set of modified equations

$$(\boldsymbol{K}_0 + \Delta\boldsymbol{K}_0)\boldsymbol{r} = (\boldsymbol{R}_0 + \Delta\boldsymbol{R}_0) \tag{3}$$

where the modified values $\boldsymbol{K}$ and $\boldsymbol{R}$ are given by

$$\boldsymbol{K} = \boldsymbol{K}_0 + \Delta\boldsymbol{K}_0 \qquad \boldsymbol{R} = \boldsymbol{R}_0 + \Delta\boldsymbol{R}_0 \tag{4}$$

Approximation of $\boldsymbol{r}$ by the CA method involves the following steps:

1) Calculate the modified values $\boldsymbol{K}$ and $\boldsymbol{R}$ by Eqs. (4).

2) Calculate the basis vectors

$$\boldsymbol{r}_1 = \boldsymbol{K}_0^{-1}\boldsymbol{R} \tag{5}$$

$$\boldsymbol{r}_i = -\boldsymbol{B}\boldsymbol{r}_{i-1} \qquad i = 2, 3, \ldots, s \tag{6}$$

where s is much smaller than the number of degrees of freedom and matrix $\boldsymbol{B}$ is defined as

$$\boldsymbol{B} = \boldsymbol{K}_0^{-1}\Delta\boldsymbol{K}_0 \tag{7}$$

Calculation of the basis vectors involves only forward and backward substitutions.

The displacements $\boldsymbol{r}$ are approximated in terms of basis vectors $\boldsymbol{r}_B = [\boldsymbol{r}_1, \ldots, \boldsymbol{r}_s]$,

$$\boldsymbol{r} = \boldsymbol{r}_B \boldsymbol{y} \tag{8}$$

where $\boldsymbol{y}$ is a vector of coefficients. Then we can write the equilibrium equations of the modified design using the preceding approximation:

$$\boldsymbol{K}\boldsymbol{r}_B\boldsymbol{y} = \boldsymbol{R} \tag{9}$$

This is an overdetermined set of n equations with s unknowns that can be solved for the vector of the unknown coefficients, $\boldsymbol{y}$, using a least-squares approach

$$\boldsymbol{y} = \left(\boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B\right)^{-1}\boldsymbol{r}_B^T\boldsymbol{R} \tag{10}$$

3) Calculate the reduced stiffness matrix $\boldsymbol{K}_R$ and load vector $\boldsymbol{R}_R$ by

$$\boldsymbol{K}_R = \boldsymbol{r}_B^T\boldsymbol{K}\boldsymbol{r}_B, \qquad \boldsymbol{R}_R = \boldsymbol{r}_B^T\boldsymbol{R} \tag{11}$$

where $\boldsymbol{r}_B$ is the $n \times s$ matrix of the basis vectors $\boldsymbol{r}_B = [\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_s]$.

4) Calculate the vector of coefficients $\boldsymbol{y}$ by solving the system

$$\boldsymbol{K}_R\boldsymbol{y} = \boldsymbol{R}_R \tag{12}$$

where $\boldsymbol{y}$ is a vector of coefficients to be determined; $\boldsymbol{y}^T = \{y_1, y_2, \ldots, y_s\}$. This $s \times s$ system is much smaller in size than the $n \times n$ original system.

5) Evaluate the final displacements by the linear combination

$$\boldsymbol{r} = y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + \cdots + y_s\boldsymbol{r}_s = \boldsymbol{r}_B\boldsymbol{y} \tag{13}$$

Leu and Huang[23] and Kirsch[17] showed that the efficiency and accuracy of the CA method for nonlinear analysis of structures improves by using Gram–Schmidt orthogonalization procedure (see Ref. 24). In this reanalysis approach, the reduced set of equations becomes uncoupled and the reduced matrix is better conditioned compared to that in normal reanalysis; thus, possible ill conditioning is avoided in nonlinear analysis. An alternative way to solve the problem of numerical instability in the CA method is by using the singular value decomposition technique to find the least-squares solution to the overdetermined set of Eqs. (9).

The CA method is different than the Neumann series method (or binomial series method). The CA method uses vectors $\boldsymbol{r}_i$ as basis [Eq. (13)], whereas the Neumann series method approximates the response of a structure merely as the sum of these vectors [Eqs. (5) and (6)]. The CA method is considerably more robust and accurate than the Neumann series method, which yields poor approximations for all but very small changes in the design variables.

III. Probabilistic Vibration Analysis of Mistuned Bladed Disk Assemblies Using the CA Method

A. Introduction

Bladed disk assemblies (BLISKs) are periodic structures exhibiting cyclic symmetry. When all of the blades are identical, identically mounted, and uniformly spaced on the disk, then the BLISK is called tuned. In real life, blades are different because of manufacturing imperfections, material tolerances, and degradation due to aging. BLISKs are imperfection sensitive systems; the dynamic response of a BLISK is extremely sensitive to variations in the properties of the blades. Indeed, even small variations in the stiffness and/or mass of the blades from their nominal values, for example, less than 1%, can change dramatically the vibratory amplitudes of the blades, for example, by more than 100%. Hence, it is important to quantify the variation in the vibrating amplitudes of a BLISK due to mistuning.

Mistuning often increases the vibratory stress levels in the blades, and it has caused many jet engine failures. Blade mistuning is not only detrimental, but instead it can be used to improve performance if a given set of mistuned blades is arranged in an optimal way, for example, by Petrov et al.[25]

Many studies have focused on mistuning of blisks aiming at predicting the vibratory response given the variability in the properties of the blades; these include studies by Whitehead,[26] Fabunmi,[27] Ewins,[28] Bendiksen,[29] Ewins and Han,[30] Griffin and Hoosac,[31] Srinivasan,[32] and Afolabi.[33] All of these studies used simplified spring–mass models for mistuning analysis. Most of the research has been limited to the deterministic aspects of blade mistuning. Ewins and Han[30] studied the importance of individual blade mistuning and the effect of the arrangement of the blades on the response of a mistuned blisk. To avoid failures due to mistuning, it is important to identify those blades with high vibration amplitudes, so that these blades could be monitored during engine tests. El-Bayoumy and Srinivasan,[34] Griffin and Hoosac,[31] Ewins and Han,[30] and Afolabi[33] investigated how to identify the blades with largest amplitudes. Afolabi showed that the highest responding blades are most likely to be those with extreme mistune. Srinivasan[32] presented a selected survey of analytical and experimental research on vibration of mistuned blisks.

Studies have focused on calculating the statistics of the vibratory amplitude of the steady-state harmonic response of BLISKs given the probability distribution of the properties of the blades. Huang[35] used an analytical approach to evaluate the mean and variance of the vibratory amplitudes of the blades. However, he did not find the probability density function of the maximum amplitude or the probability of the maximum amplitude of any blade exceeding a critical value. Sinha[36] calculated the statistics of the forced response of a BLISK. However, his method was applicable for only small amounts of mistuning in the system. Srinivasan[32] used Monte Carlo simulation to evaluate the forced response of BLISKs. Hamade and Nikolaidis[37] studied the sensitivity of the statistics of the forced response of nearly periodic structures to the degree of mistuning. Various methods have been proposed to compute the forced response of BLISKs. Griffin and Hoosac[31] proposed an efficient algorithm for calculating the statistics of the blade responses of a BLISK and performed a statistical investigation of the mistuning.

Recent studies on probabilistic analysis of mistuned BLISKs use realistic finite element models of BLISKs instead of lumped mass–spring models. This requires simulating hundreds or thousands of randomly mistuned BLISKs and running a finite element code for each realization. This involves a high computational effort, and therefore, there is a need for new efficient tools for predicting the statistics of the forced response. Yang and Griffin[38] developed a reduced-order model, which uses a subset of the nominal system modes (SNM), to predict the steady-state response of mistuned BLISKs. In the SNM approach, the blade variations are accounted for by variation in the elastic modulus of the blades, assuming that the system stiffness matrix is changed proportionally, and therefore, the mistuned modes are linear combinations of the tuned modes. More recently, Castanier et al.[39] developed a reduced-order modeling technique, which is based on the component modes of vibration calculated from a finite element model, for calculating the response statistics of mistuned turbomachinery rotors.

In the past few years, the focus shifted to probabilistic modeling of mistuning including the stochastic spatial variation of the blade thickness due to manufacturing imperfections. Ghiocel[40,41] indicated that because of random manufacturing deviation, the system stiffness and mass matrices suffer nonproportional variations that can affect the mistuned response, especially when mode families with closely spaced frequencies interact. Ghiocel suggested the CA method as a potential candidate method for predicting the response of BLISKs in such situations [Ghiocel, D. M., "Mistuning Analysis of Bladed-Disks," BLADE-GT Engineering Research Report, STI Technologies, for U.S. Air Force Research Lab./PRT, Wright–Patterson Air Force Base, Dayton, OH, June 2002, Chap. 3 (unpublished, contains proprietary information)].

We want to show that the CA method is an efficient tool for performing probabilistic analysis of mistuned BLICKs. Because even small changes in the properties of the blades can change dramatically their vibratory responses, analysis of a mistuned BLISK is a good problem for testing the CA method.

B. Probabilistic Analysis Procedure

Vibration analysis of mistuned BLISKs is categorized into deterministic and probabilistic. In deterministic analysis, we change the properties of the blades and calculate the response of the perturbed structure. Probabilistic analysis predicts the probability distribution of the vibratory amplitudes of the blades due to mistuning.

We developed two procedures that approximate the steady-state harmonic response of a BLISK as a function of the random properties only and as a function of both these properties and the excitation frequency. The response is computed at equally spaced frequencies in a certain frequency range close to resonance using the CA method. Then the maximum vibratory amplitude of each blade over a frequency range is found. The probability distribution of this maximum amplitude is estimated using Monte Carlo simulation.

Table 1 Values of model parameters

Parameter	Value
m_1	0.0114 kg
m_2	0.0427 kg
m_3	0.0299 kg
k_1	430,300 N/m
k_2	17,350,000 N/m
k_3	7,521,000 N/m
k_6	30,840,000 N/m
c	0.27628 N · s/m
Excitation force f_0	10,000 N

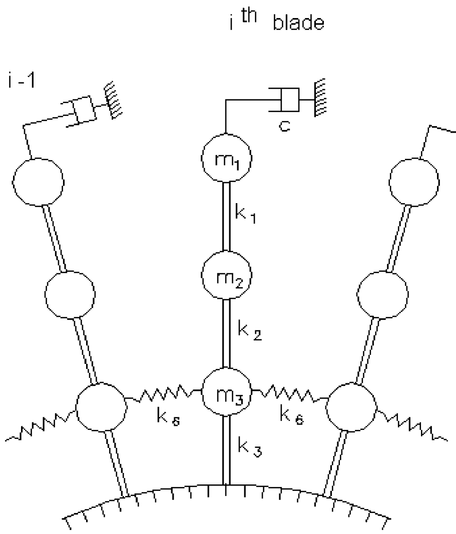


Fig. 1 Spring–mass model.

1. BLISK Model Description

A model of a BLISK with 72 blades consisting of a lumped mass–spring–damper system is shown in Fig. 1. Griffin and Hoosac[31] employed this model for their statistical investigations on turbine blade mistuning. This model is linear. It accounts for the structural coupling between the blades and considers aerodynamic damping as a source of dissipation of energy. The parallel lines indicate beam-like elements that have zero extensional flexibility and transverse stiffness. The model parameters are shown in Table 1. A zero-order (blade forces in phase) harmonic excitation is applied on the BLISK.

The governing equations of motion for the ith blade and disk segment are

$$m_1\ddot{x}_{1i} + k_1(x_{1i} - x_{2i}) + c\dot{x}_{1i} = f_0 \exp[j(\omega t + H2\pi i/N_b)] \tag{14}$$

$$m_2\ddot{x}_{2i} + k_1(x_{2i} - x_{1i}) + k_2(x_{2i} - x_{3i}) = 0 \tag{15}$$

$$m_3\ddot{x}_{3i} + k_6\left(2x_{3i} - x_{3(i+1)} - x_{3(i-1)}\right) + k_2(x_{3i} - x_{2i}) + k_3x_{3i} = 0 \tag{16}$$

where f_0 is the amplitude of the harmonic force applied to the ith blade, ω is the frequency, and $H2\pi i/N_b$ is the phase angle of the ith blade for an excitation order H, which is zero in this study. N_b is the number of blades. The remaining symbols are explained in Fig. 1. When the preceding equations are assembled for each blade, the equations of motion for the steady-state forced harmonic response of the whole BLISK in the frequency domain can be written as

$$[\boldsymbol{K} - \omega^2\boldsymbol{M} + \boldsymbol{C}\omega j]\boldsymbol{X} = \boldsymbol{F} \tag{17}$$

or

$$\boldsymbol{D}(\omega)\boldsymbol{X} = \boldsymbol{F} \tag{18}$$

where $\boldsymbol{D}(\omega) = [\boldsymbol{K} - \omega^2\boldsymbol{M} + \boldsymbol{C}\omega j]$ is the dynamic stiffness matrix.

In the nominal model all of the blades are identical. That is, the stiffnesses of all of the blades, represented by springs k_1 and k_2, are equal. This is a tuned system, and the resonant amplitudes of the blades for such a system are equal, when the applied forces on the blades have equal amplitudes and they are in phase. To simulate mistuning, the structural properties of the blades are perturbed by changing the spring stiffnesses k_1 and k_2 to $k_1(1+\varepsilon_1)$ and $k_2(1+\varepsilon_2)$, where ε_1 and ε_2 are normal random variables with zero mean and standard deviation equal to 1%. For every blade, random variables ε_1 and ε_2 are assumed equal. The variation in the stiffness of one blade is independent of the variation in the stiffness of the other blades. The equations of motion for the perturbed BLISK are

$$\boldsymbol{K}' = \boldsymbol{K} + \Delta\boldsymbol{K} \tag{19}$$

$$[\boldsymbol{K} + \Delta\boldsymbol{K} - \omega^2\boldsymbol{M} + \boldsymbol{C}\omega j]\boldsymbol{X}_p = \boldsymbol{F} \tag{20}$$

or

$$\boldsymbol{D}'(\omega)\boldsymbol{X}_p = \boldsymbol{F} \tag{21}$$

$\boldsymbol{X}_p$ is the response of the perturbed model, $\boldsymbol{K}'$ is the perturbed stiffness matrix, and $\boldsymbol{D}'(\omega)$ is the perturbed dynamic stiffness matrix.

The response of the nominal and perturbed BLISKs can be computed by direct inversion of the dynamic stiffness matrix. This calculation is performed for every frequency and every realization of a BLISK in a Monte Carlo simulation for probabilistic analysis of a mistuned BLISK. In the following paragraphs, we present the two CA-based approaches that approximate the response as a function of 1) the random properties only and 2) both the random properties and the frequency.

2. Approximating Response as a Function of Random Properties

Here we approximate the response of the perturbed BLISK at a particular frequency, $\boldsymbol{X}(\omega, \boldsymbol{P} + \Delta\boldsymbol{P})$, in terms of the response of the nominal BLISK at the same frequency, $\boldsymbol{X}(\omega, \boldsymbol{P})$, and the change in the properties, $\Delta\boldsymbol{P}$. At every frequency, the response of the nominal BLISK is computed by the direct method and the response of the perturbed design is approximated by the CA method. This procedure is repeated for evaluating the response at other frequencies in the desired frequency range. Here the change in the dynamic matrix is due to the perturbation in the properties of the BLISK only:

$$[\boldsymbol{K} + \Delta\boldsymbol{K} - \omega^2\boldsymbol{M} + \boldsymbol{C}\omega j]\boldsymbol{X}_p = \boldsymbol{F} \tag{22}$$

This approach requires n complete analyses of a BLISK in a Monte Carlo simulation, where n is the number of frequencies at which the response is computed.

3. Approximating Response as a Function of Both Random Properties and Frequency

Here we approximate the response of a perturbed BLISK at a frequency $\omega + \Delta\omega$, $\boldsymbol{X}(\omega + \Delta\omega, \boldsymbol{P} + \Delta\boldsymbol{P})$, in terms of the response of the nominal BLISK at frequency, ω, $\boldsymbol{X}(\omega, \boldsymbol{P})$, where $\boldsymbol{P}$ is the vector of the random properties of the nominal blisk. The response of the nominal BLISK is obtained at a single frequency ω_b by the direct method, and then the response of a perturbed BLISK at other frequencies is approximated by the CA method. The change in the dynamic matrix of the BLISK is due to both the change in the frequency and in the blisk properties:

$$[\boldsymbol{K} + \Delta\boldsymbol{K} - (\Delta\omega + \omega)^2\boldsymbol{M} + \boldsymbol{C}(\Delta\omega + \omega)j]\boldsymbol{X}_p = \boldsymbol{F} \tag{23}$$

This approach requires only one complete analysis of a BLISK for the entire Monte Carlo simulation, whereas the approach that approximates the response as a function of the properties only requires a complete analysis for each frequency of interest. On the other hand, it is harder to approximate the response as a function of both the frequency and the properties than as a function of the properties only.

4. Finding the Number of Basis Vectors in Monte Carlo Simulation

One should determine the number of basis vectors for the approximation of the response. If too few basis vectors are used, then the approximation could be inaccurate. On the other hand, we should not include linearly dependent vectors into the basis because they cause numerical difficulties in finding the response. The following procedure was used to select the number of basis vectors to be used in simulation:

1) We generated 10 random realizations of the BLISK.

2) For each realization, we estimated the steady-state harmonic response of the BLISK starting with two basis vectors and incrementing the number of basis vectors by one. We stopped when the response was close (error less than 5%) to the response calculated by using inversion of the dynamic stiffness matrix in a frequency range around resonance.

3) We used the maximum number of basis vectors for the 10 BLISKs in Monte Carlo simulation.

We should avoid using linearly dependent basis vectors in the approximation of the response, in each simulation cycle. For this purpose, when we solved the overdetermined system of equations (9) using singular value decomposition, we checked whether matrix $\boldsymbol{Kr}_B$ had singular values. If zero values were found, the number of basis vectors was reduced until a matrix with no singular values was obtained in Eq. (9).

C. Numerical Example

1. Free Vibration Analysis

The first natural frequency of the nominal BLISK was found equal to 919.90 Hz. The first group of frequencies of a mistuned BLISK consisted of 72 frequencies ranging from 919 to 962 Hz. The BLISK had higher vibrating amplitudes in the first vibration mode than in other modes. Hence, a frequency range of 900–1000 Hz was selected, and the response was obtained in this frequency range with a step size of 0.5 Hz. Figures 2 and 3 show the mode shapes for the first mode (corresponding to the first natural frequency of the system) for the nominal BLISK (with all blades identical) and the perturbed BLISK, respectively. In Figs. 2 and 3, the X axis represents the amplitudes of the blades, the Y axis represents the three masses, and the Z axis represents the 72 blades. Figures 2 and 3 show that even a 1% variation in the stiffness of the blades from their nominal values changes dramatically the first mode shape of the BLISK. All of the blades of the nominal BLISK vibrate in phase and have equal amplitudes. When the blade properties are perturbed randomly from their nominal values, the blades vibrate with considerably different amplitudes. The mode shape of the perturbed system is localized in the sense that vibration is confined in a relatively small part of the structure. This phenomenon, in which the mode shapes of a nearly periodic structure become localized, is called mode localization.

2. Steady-State Harmonic Response of the BLISK by Direct Inversion and the CA Method

Figure 4 shows the response of the nominal and perturbed BLISKs obtained by inversion of the dynamic matrix. The response obtained by direct inversion of the dynamic matrix is considered the true one. Even a small variation in the stiffnesses of the blades causes a large variation in the dynamic response. Specifically, a variation in the blade stiffnesses with 1% standard deviation results in a variation of the maximum response within the range from 4×10^{-4} to 10×10^{-4} m. Moreover, the variation in the blade stiffness increases the maximum vibratory amplitude of the blades from 6×10^{-4} (for the tuned BLISK) to 10×10^{-4} m. Because of this great sensitivity of the response to imperfections of the blades, the mistuning problem is a very good example with which to test the CA method.

Figure 5 shows the error in the approximation of the response as a function of the random parameters of the blades only. It is important to predict accurately the response in the frequency range from 900 to 930 Hz because the maximum response occurs in this range. Four basis vectors were used in the approximation. The error is smaller than 1% of the true response between 900 and 930 Hz and less that 2% between 900 and 950 Hz.

We also approximated the response as a function of both the random parameters of the blades and the frequency. For this purpose, the true response of the tuned BLISK was calculated at the resonance frequency. Figures 6a and 6b show the error in the approximation of the response with four and seven basis vectors, respectively. Both approximations are accurate in the frequency range in which the

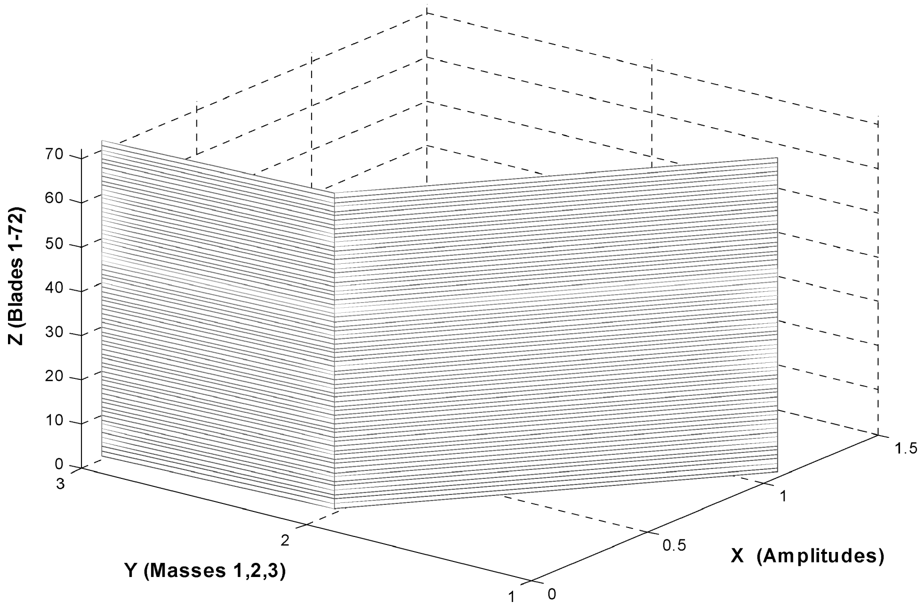


Fig. 2 First mode shape of tuned BLISK.

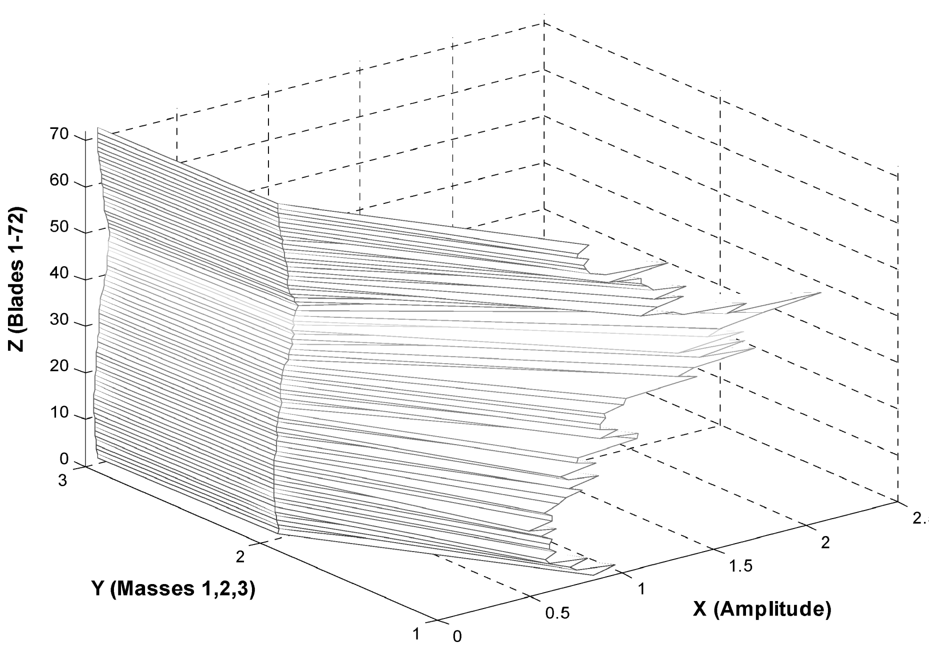


Fig. 3 First mode shape of mistuned BLISK.

response is maximum. The error in both approximations is less than 5% in the range from 900 to 940 Hz. The approximation with seven basis vectors has lower error than the approximation with four basis vectors in the range from 930 to 940 Hz. The error is several times higher than the error when the response is approximated as a function of the random properties only in frequencies far away from the resonant frequency of the tuned blisk, but the error in these frequencies is not important.

3. *Forced Response Statistics*

Statistical investigations involve computing the response for hundreds or thousands of simulated mistuned blisks. In this study, 1000 BLISKs were simulated, and a histogram of the peak blade amplitudes was plotted using the CA method and compared to the histogram obtained by the direct method. Peak blade amplitude is the maximum amplitude of each blade over the range from 900 to 1000 Hz. For one realization of a BLISK, there are 72 amplitudes, one for each blade. Thus, the histograms are based on 72,000 peak amplitudes. Figures 7 and 8 show the histograms of the amplitudes of the blades obtained by the direct and the CA methods, respectively. In the CA method, the response was approximated as a function of both the random properties and the frequency. The histograms and the statistics of the amplitudes obtained by the two methods agree very well. Specifically, the errors in the minimum and maximum responses and the mean and standard deviation of the response computed by the CA method were always less than 1% of the true value of the response.

The sample standard deviation of the estimated (mean) probability of the response of a blade, p, being in a particular bin of the histogram is $\sqrt{[p(1-p)/N]}$, where N is the number of response amplitudes used to estimate this probability (72,000). We can calculate the standard deviation of the frequency for the particular bin by multiplying the standard deviation of the estimated probability by N. Thus, the standard deviation of the frequency in each bin of the histogram f is $\sqrt{[f(N-f)/N]}$. In Figs. 7 and 8, the most likely

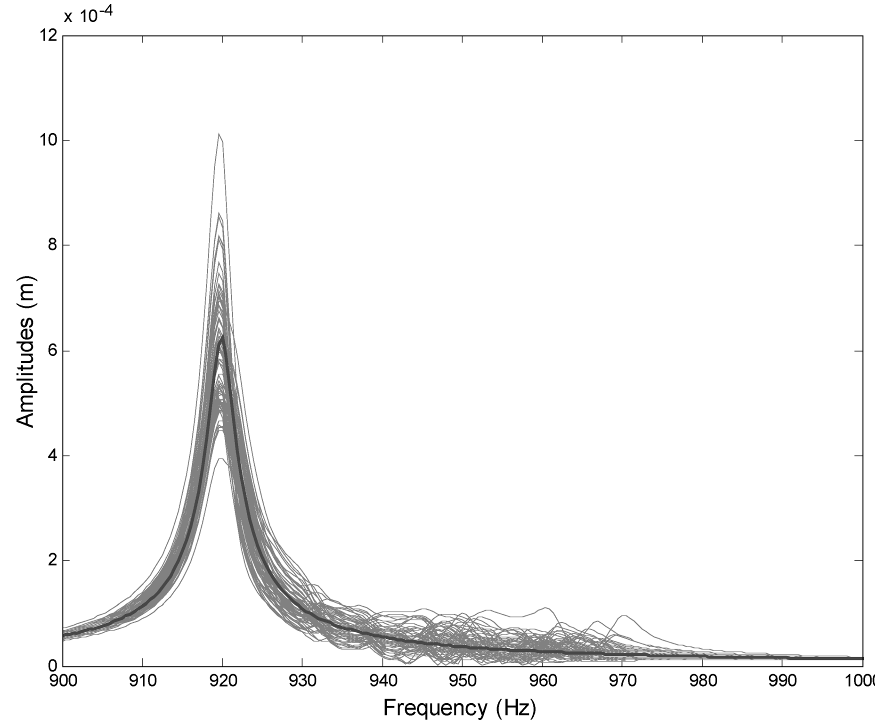


Fig. 4 Response of tuned BLISK (solid black line) and mistuned BLISK (gray lines) calculated by direct matrix inversion.

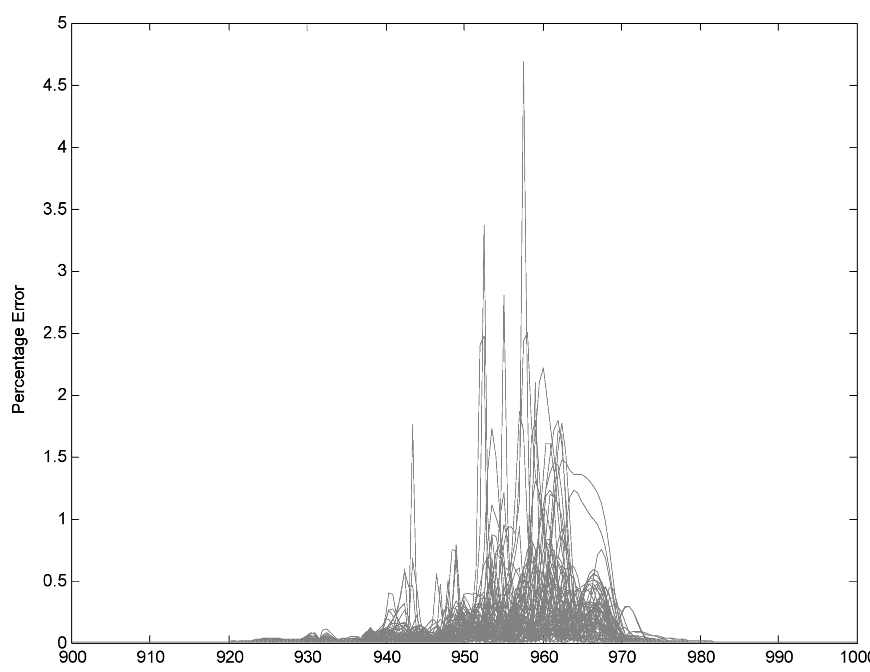


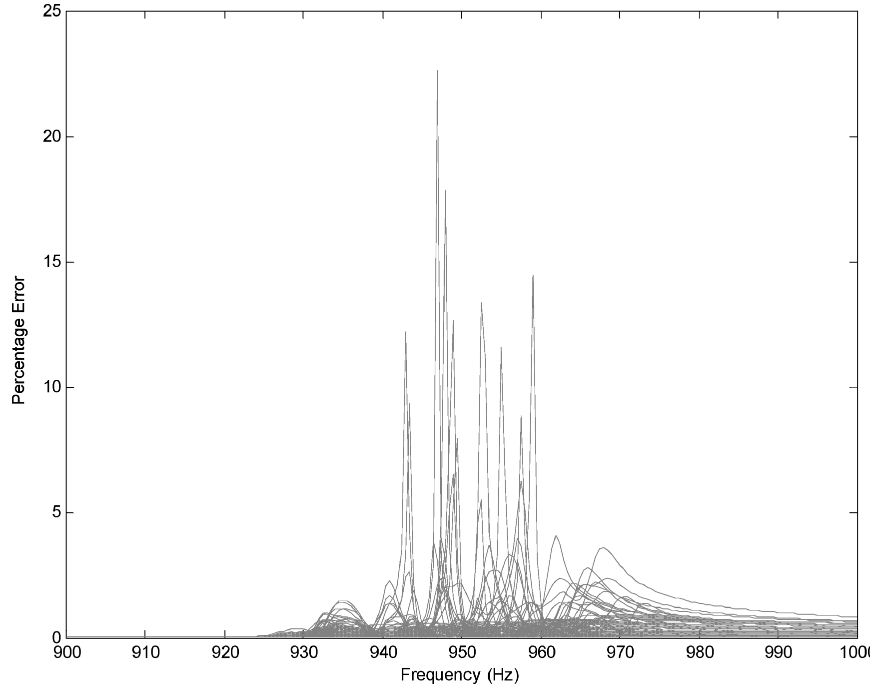
Fig. 5 Percent error in response of mistuned BLISK; response approximated as function of random properties only using four basis vectors.

value of the amplitude is about 6×10^{-4} m, and the frequency of the bin corresponding to this amplitude is 4500. The estimated frequency is normally distributed according to the limit-state theorem. The standard deviation for this frequency is 65, which means that the true frequency is in the range from 4435 (estimated frequency minus its standard deviation) to 4565 (estimated frequency plus its standard deviation) with probability 68%.

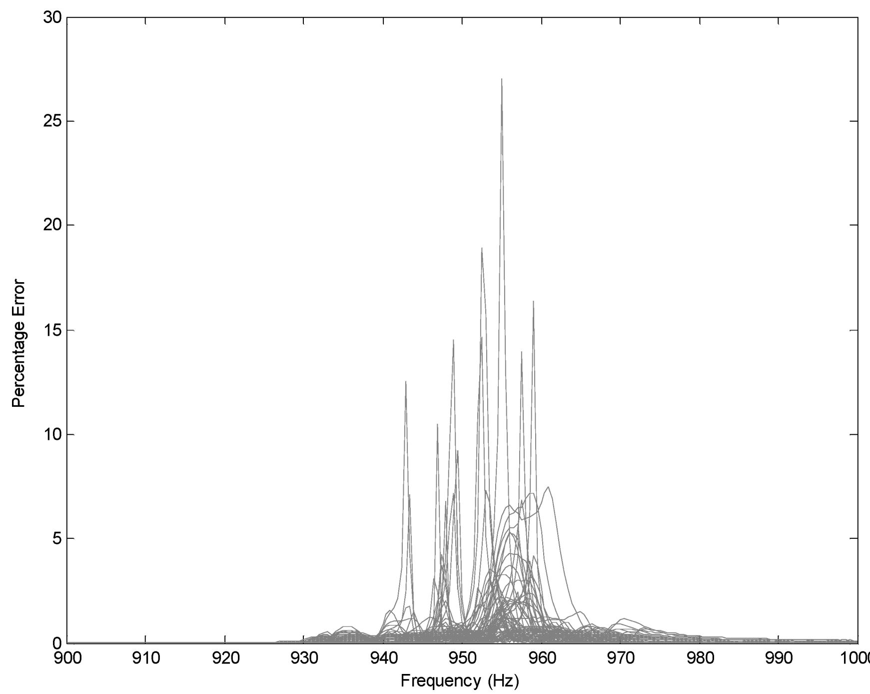
The standard deviation of the mean value of the response is $\sigma/\sqrt{N}$, where σ is the standard deviation of the response. The standard deviation of the standard deviation is $\sigma/\sqrt{(2N)}$. Figures 7 and 8 also show the 68% confidence intervals of the mean values and standard deviations of the response amplitudes estimated from Monte Carlo simulation.

4. *Computational Considerations*

The BLISK has 216 degrees of freedom, and hence, evaluating the response of the BLISK by the direct method involves inverting a 216×216 complex matrix. On the other hand, the CA method involves only computing the basis vectors and then, depending on the number of basis vectors, inverting a reduced complex matrix. The reduced matrix would have a maximum of 10–15 degrees of freedom. The response of the perturbed BLISKs can be approximated



a)



b)

Fig. 6 Percent error in response of mistuned blisk approximated as function of both random properties and frequency: a) four and b) seven basis vectors.

by the CA method either as a function of the random properties only or as a function of both the random properties and frequency. In the former approximation, one complete analysis is required at each frequency, whereas in the latter only one complete analysis is required.

A summary follows of the amount of computations required for direct method and CA method if the response of 1000 realizations of a BLISK design is to be obtained at n frequencies:

1) Direct method involves $1000 \times n$ direct inversions of a 216×216 complex dynamic matrix.

2) The CA method that approximates the response as a function of the random properties only involves 1000 direct inversions $+ n \times 1000$ approximations (steps 1–5 in the description of the CA method).

3) The CA method that approximates the response as a function of both the random properties and frequency involves 1 direct inversion $+ [(n-1) + (n \times 1000)]$ approximations.

A summary follows of the computational time required for computing the response of the BLISK. All of the analyses were done using a FORTRAN 90 program on an Intel Pentium III processor, 728-MHz machine. Table 2 shows the CPU times for computing the response of the BLISK by dynamic stiffness inversion and the CA method. Note that, in the spring mass model used for analysis, the stiffness matrix is sparse and sparseness

Table 2 Summary of CPU times for probabilistic analysis of BLISK[a]

Method	Time, min
Direct	1412.90
CA	
4 Basis vectors	58.73
5 Basis vectors	71.76
6 Basis vectors	83.00

[a]CPU times for calculating response of nominal model and perturbed model for 1000 simulations (frequency range 910–930 Hz).

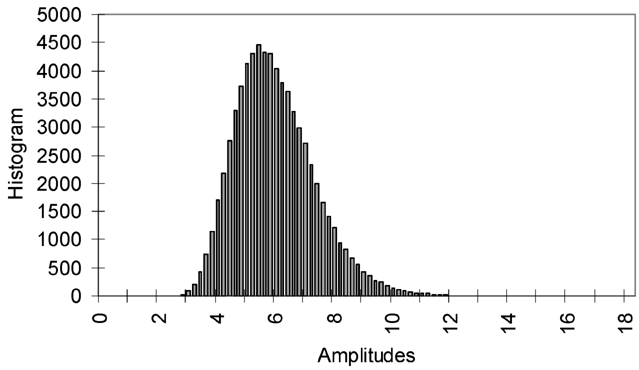


Fig. 7 Histogram of blade amplitudes direct method: maximum 16.0497, mean 5.86994 (5.86485, 5.87503), minimum 2.4116, and standard deviation 1.36535 (1.36175, 1.36895). Numbers in parentheses show confidence intervals containing these values with probability 0.68.

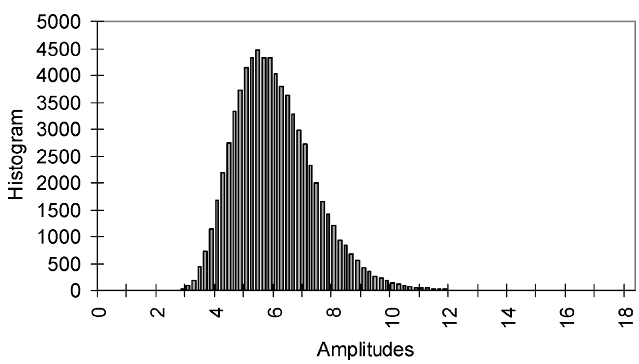


Fig. 8 Histogram of blade amplitudes, CA method, with response approximated as function of both random properties and frequency: maximum 16.0997, mean 5.87043 (5.86535, 587551), minimum 2.4334, and standard deviation 1.36444 (1.36084, 1.36804). Numbers in parentheses show confidence intervals containing these values with probability 0.68.

of the matrix was not considered while computing the response of BLISK.

5. Effect of the Stiffness of the Disk of a BLISK on the Maximum Response of the Blades

When the stiffness of the disk of a BLISK increases, the coupling between the blades reduces. It is known that in a mistuned almost periodic system, such as the BLISK, localization of the modes becomes more pronounced when the coupling between its subsystems (in our case, the blades) reduces. Keerti[42] observed that the maximum response of the blades of a BLISK changes in a nonmonotonic fashion with the degree of mistuning (the standard deviation of variables ε_1 and ε_2) and the degree of coupling between the blades (measured by the disk stiffness). For values of the disk stiffness close to the nominal value of a real life system, the maximum response increases with the degree of mistuning. However for large values of the disk stiffness, the response of the blades initially increases with the degree of mistuning up to a certain value and then decreases for higher values. Moreover, the variability in the response tends to decrease with the blade stiffness. Therefore, for BLISKs with very high disk stiffness, mistuning could have a beneficial effect on reducing the vibratory responses of the blades.

6. Observations from the Mistuning Analysis

1) Because a BLISK is an imperfection sensitive system, approximating its response is a challenging task. The CA method estimated the response of the BLISKs considered in this study with almost the same accuracy as a more expensive method that uses direct dynamic matrix inversion.

2) The CA method estimated the probability density function of the blade amplitudes of a BLISK with variability in its blade properties accurately.

3) The computational effort involved in computing the response using the CA method of the BLISK is considerably lower than that of a method that inverts the dynamic matrix.

IV. Progressive Collapse Analysis of Truss Structures Using the CA Method

A. Introduction

Redundant systems can survive even if one or more components fail. Failure of many redundant structural systems involves progressive collapse of their components. Examples of systems that exhibit progressive collapse are offshore platforms, bridges, frames, and automotive structures in crash.

Progressive failure analysis of a structure involves analyzing the response of the structure due to failure or damage of one or more of its components. The loss or failure of one or more components causes redistribution of the forces in the remaining structural components and may lead to collapse.

The performance of such a system is sensitive to the variability in the properties of the components. Even small changes in the location or properties of a component can change the sequence of collapse, which can change significantly the ultimate collapse load and absorbed energy. Uncertainties in the material properties or geometry of the components of the system lead to uncertainties in the collapse load and the energy absorbed in failure. Hence, it is important to quantify the effect of these uncertainties on the collapse load using probabilistic analysis.

Calculation of the collapse load requires analysis of a sequence of damaged structures obtained from the intact structure when its members fail. Quantification of the uncertainty in the collapse load involves estimation of the probability distribution of the collapse load of the damaged structure at different stages of the collapse process. The probability distribution of the collapse load is estimated using Monte Carlo simulation, and this involves repeated analyses of the damaged structures. A challenge in developing tools for quantifying the uncertainty in the collapse load is the high computational cost. Here we employ the CA method to probabilistic analysis of systems with progressive collapse.

B. Progressive Failure of Trusses

In this study, the efficiency and accuracy of the CA method is demonstrated on the progressive collapse analysis of a 50-bar truss (Fig. 9). In progressive collapse analysis of trusses, the truss is loaded incrementally until the first element failure occurs. When an element fails, the applied load is redistributed among the other elements of the truss. The load is further increased incrementally until another element fails. This process continues until the structure becomes a mechanism.[42] The load at which the structure becomes a mechanism is the ultimate collapse load.

To obtain the probability distribution of the collapse load, random realizations of the truss structure are generated by varying the yield stresses of all of the elements of the truss according to their probability distributions and the collapse loads of these realizations are calculated.

C. Numerical Results

Table 3 specifies the material properties and the dimensions of the truss. In this study, the probability distribution of the yield stress

Table 3 Structural properties and geometry of 50-bar truss

Property	Value
Young's modulus E	10,000 ksi (6.89×10^{10} Pa)
Cross-sectional area	1 in.^{2} (6.45×10^{-4} m^2)
Length	
Elements 1–30	100 in. (2.54 m)
Elements 31–50	141 in. (3.58 m)
Yield stress	35 ksi (2.41×10^8 Pa)

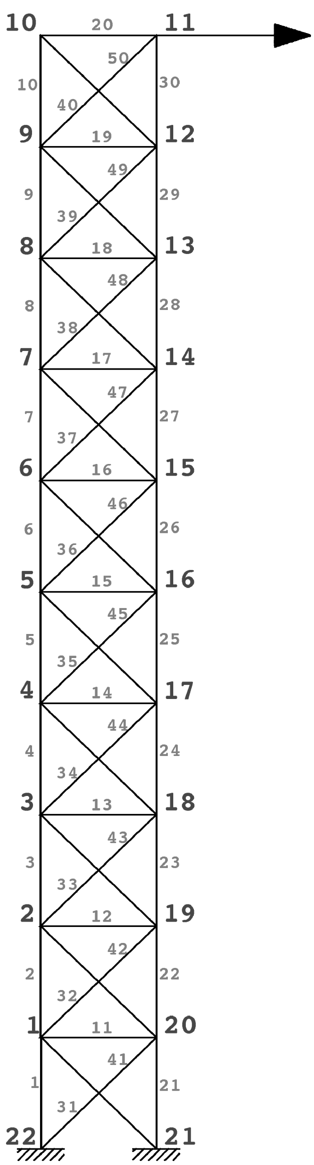


Fig. 9 Truss with 50 bars.

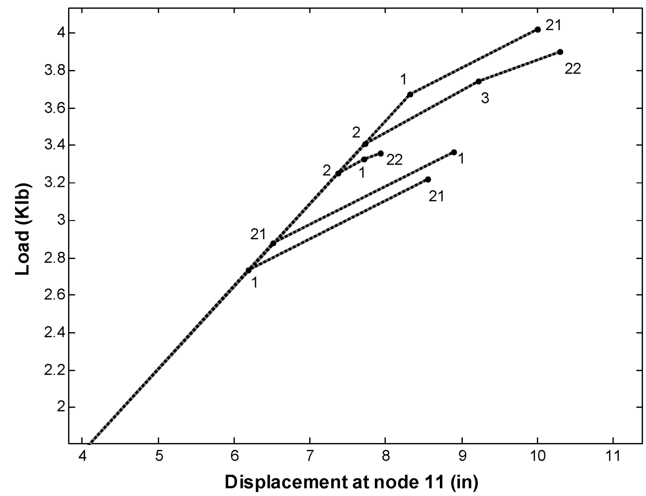


Fig. 10 Progressive failure modes in 50-bar truss.

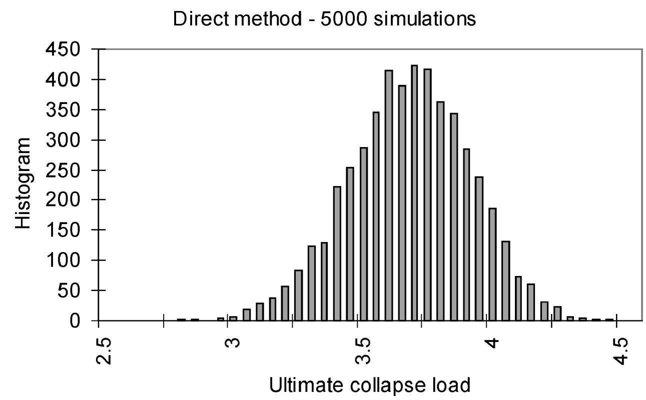


Fig. 11 Ultimate collapse load statistics for 50-bar truss by CA method: maximum 4.4202, mean 3.655, minimum 2.7048, and standard deviation 0.2373.

in each member is assumed normal with a standard deviation of 10% of the mean value. The yield stresses of the members are mutually independent. Ductile failure is assumed for all of the truss elements, and the material behavior is assumed perfectly elastic–plastic. A horizontal force was applied on the top of the truss as shown in Fig. 9. The following assumptions are made for the truss analysis: 1) a truss consists of bars that can only take tensile or compressive loads, 2) all loads and reactions are applied at the joints only, and 3) members are connected together at their ends by frictionless pin joints. A summary follows of the results on the 50-bar truss.

1) Figure 10 shows the sequence of member failures of five realizations, which indicates the progressive failure behavior in trusses. Note that the variation in the structural properties changes the failure sequence and the ultimate collapse load.

2) Figure 11 shows the histogram of the ultimate collapse load of the truss obtained by the CA method. The results of the direct method were practically identical with the results of the CA method. Therefore, they are not shown.

Note that the example of the 50-bar truss considered is favorable for the approximation of the collapse load because collapse occurs after two or three member failures. The collapse load in this case can be calculated by the Woodbury formula using only two basis vectors.

D. Computational Considerations

With the assumption that collapse of a truss in the Monte Carlo simulation approach involves M bar failures, the entire analysis to determine the collapse load needs to be repeated M times. If the number of degrees of freedom for the truss structure is n, direct analysis requires inversion of $2M(n \times n)$ matrices.[42] This means that, for N simulations, $N \times 2M(n \times n)$ matrices are inverted in direct analysis. When the CA method is used, only one $(n \times n)$ matrix is inverted and approximately $N \times 2M$ reduced matrices are inverted [Eq. (11)].

For instance in the 50-bar truss example, assuming 3 failure modes for each simulated truss and 5000 simulations, the direct method inverts 30,000 (40×40) matrices and the CA method inverts one 40×40 matrix and about 30,000 2×2 matrices, assuming that it uses two basis vectors.

V. Conclusions

This paper presented an efficient and accurate tool for probabilistic analysis of structures using the CA method. The paper illustrated the efficiency and accuracy of this method in two engineering problems. The first problem involved a BLISK assembly whose performance is sensitive to variability in the input parameters. The second problem involved a redundant structure with progressive collapse failure modes. The following observations and conclusions were drawn:

1) The CA method estimated accurately the responses of the BLISK assemblies considered despite the sensitivity of the response to the random properties.

2) Progressive failure analysis of redundant structures, such as offshore platforms and cars in crash, involves finding the probability distribution of the collapse load and the energy absorbed during failure. In the examples considered, the CA method approximated the response of the structures accurately even when their topology changed considerably due to failure of some of its members. The CA method enabled the authors to perform only one direct analysis of a nominal structure and approximate the responses of tens of thousands of partially damaged structures, whose material properties and topology differed significantly from those of the nominal structure, in Monte Carlo simulation. However, note that predicting the collapse load of the truss is easy for the CA method because the collapse of the truss involves failure of two or three bars and the ultimate collapse load can be exactly calculated by the Woodbury formula.

3) In both example problems considered, the CA method enabled us to reduce considerably the computational cost.

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